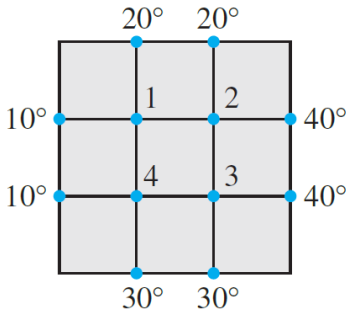
Exercise 33

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes— to the left, above, to the right, and below.[2] For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4 \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



Write a system of four equations whose solution gives estimates for the temperatures $T_1, \ldots, T_4$.

Solution

The temperature at each node is the average of the temperatures around it.

$$\text{Node 1:} \quad T_1 = \frac{10 + 20 + T_2 + T_4}{4} \quad \rightarrow \quad 4T_1 - T_2 - T_4 = 30$$

$$\text{Node 2:} \quad T_2 = \frac{20 + 40 + T_1 + T_3}{4} \quad \rightarrow \quad 4T_2 - T_1 - T_3 = 60$$

$$\text{Node 3:} \quad T_3 = \frac{30 + 40 + T_2 + T_4}{4} \quad \rightarrow \quad 4T_3 - T_2 - T_4 = 70$$

$$\text{Node 4:} \quad T_4 = \frac{10 + 30 + T_1 + T_3}{4} \quad \rightarrow \quad 4T_4 - T_1 - T_3 = 40$$

[2] See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.